Hybrid finite difference schemes from operator splitting for solving Burgers' equation

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Burgers' equation appears as a model in turbulence and gas dynamics. We develop hybrid finite difference methods resulting from operator splitting for solving it. Among the hybrid finite difference methods developed are the Crank–Nicholson–Lax–Fredrichs, Crank-Nicholson–Du Fort and Frankel and Crank–Nicholson–Lax-Fredrich–Du Fort and Frankel. We determine that Lax–Fredrichs method reduces the efficacy of the Crank–Nicholson method whereas the Du Fort and Frankel method increases the efficacy of this method (the Crank-Nicholson method). The Du–Fort and Frankel method actually increase the number of grid points involved. The increase in number of grid points used is responsible for the improved accuracy of the pure Crank–Nicholson and the hybrid Crank–Nicholson-Lax–Fredrichs’ schemes. The hybrid Crank-Nicholson-Lax-Friedrichs’ scheme is the most accurate.

Keywords: Burgers' equation, operator splitting, Crank-Nicholson method.

Introduction

Finite difference methods for solving the heat equation

Consider the heat equation:

\[ u_t = \alpha u_{xx} \quad u(0, x) = u_0(x) \quad (1.1.1) \]


Schmidt and Lax-Fredrichs’ methods are conditionally stable whereas Crank-Nicholson and Du Fort and Frankel are unconditionally stable.

In our paper we develop blended (hybrid) finite difference methods for finding the numerical solution of the Burgers’ equation

\[ u_t + \beta u u_x = \alpha u_{xx} \quad (0 \leq x \leq 1) \times (t \geq 0) \quad (1.1.2) \]
\[ u(0,x) = u_0(x) \]
\[ u_x(0,t) = p(x) \]
\[ u_x(1,t) = q(x) \]
resulting from the operator splitting.

**Overview of Operator Splitting**

The operator splitting technique for the linear parabolic equation is outlined, thus:

\[ u_t = Lu, \quad (a \leq x \leq b) \times (0 \leq t \leq T) \]  
\[ u(x,0) = u_0(x) \]  

where \( u = u(x,t) \) and \( L \) is a linear differential operator.

Consider the Taylor's expansion

\[ u(x,t+k) = u(x,t) + k \frac{\partial}{\partial t} u(x,t) + \frac{k^2}{2!} \frac{\partial^2}{\partial t^2} u(x,t) + ... \]

\[ = \left( 1 + k \frac{\partial}{\partial t} + \frac{k^2}{2!} \frac{\partial^2}{\partial t^2} + ... \right) u(x,t) \]

\[ = e^{k \frac{\partial}{\partial t}} (u(x,t)) \]  

(1.2.3)

In equation (1.2.3) \( \frac{\partial}{\partial t} \) can be replaced by \( L \) that is

\[ u(x,t+k) = e^{ktL} u(x,t) \]  

(1.2.4)

The exact solution of the equation (1.2.1)-(1.2.2) at the grid point \( (x = mh, t = nk) \) is \( u(x,t) \). The approximate solution at this point is denoted by \( U_{mn} \). Equation (1.2.4) can be written as;

\[ U_{m,n+1} = e^{ktL} U_{m,n} \]  

(1.2.5)

In equations (1.4) and (1.2.5) \( e^{ktL} \) is called the solution operator for equation (1.2.1) \( L \) is replaced by finite difference approximation. In equation (1.2.5) \( L \) can be taken to be a sum of differential operators with respect to \( x \). If

\[ L = L_1 + L_2 + L_3 + ... + L_s = \sum_{i=1}^{s} L_i \]

then equation (1.2.5) can be written as

\[ U_{m,n} = e^{k \sum_{i=1}^{s} L_i} U_{m,n} \]

\[ = (e^{k(L_1+L_2+L_3+...+L_s)}) U_{m,n} \]  

(1.2.6)
The approximate solution can be obtained from equation (1.2.7) by first solving $U^{(S)}_{m,n+1} = e^{kL_S} U_{m,n}$ and then using this solution it is found that $U^{(S-1)}_{m,n+1} = e^{kL_{S-1}} U_{m,n}$

This continues until $U^{(1)}_{m,n+1}$ is obtained which is actually the approximate solution of equation (1.2.1) (see Istvan, 2003).

For the Burgers equation

$$u_t + \beta uu_x = \alpha u_{xx}$$

The number of operators are two; that is $S=2$ with

$$L_1 = -\frac{\beta}{h} U_{m,n} \delta_x$$

and

$$L_2 = \frac{\alpha}{\delta_x^2} \delta_x^2$$

The discretizations of $L_1$ and $L_2$ can now be done using various ways.

Historical background of operator splitting techniques can now be provided. Koller and Krylov (2006) demonstrated and discussed time integration due to operator splitting for linear 1-D parabolic equations. Ames (1977) and Mitchell and Griffiths (1980) describes additive operator splitting for parabolic equations which are more than one dimensional and were developed by Yanenko and Marchuk. Yanenko splitting is called first order operator splitting. Another splitting method mentioned by the same Mitchell and Griffiths (1980) which is called second order was developed by Strang in the 1960s. Istvan gives an elaborate discussion of operator splitting for parabolic equations. Le Veque and Oliger (1983) describes additive operator splitting for hyperbolic partial differential equations. Splitting method has been used by Hvistendahl (1997) and Evje et al., (1998) to find the numerical solution of convection–diffusion equation.

In the study, a hybrid finite difference schemes based on Yanenko and Marchuk ideas to solve the nonlinear Burgers’ equation was developed. Solution of the heat equation were found and discussed.

Below, hybrid finite difference schemes resulting from first order operator splitting are developed.

**Hybrid Finite Difference Schemes from Operator Splitting**

The word hybrid means blended or “marrying” two or more ordinary finite difference methods. To begin with, it is necessary to develop the pure Crank-Nicholson scheme as follows:

**First order operator splitting**

The first order operator splitting numerical solution is given by
\[ U_{m,n+1} = \left(e^{\Delta t_1} e^{\Delta t_2}\right) U_{m,n} \]  

where \( L_1 \) and \( L_2 \) are the operators stated above, then equation (2.1.1)  
\[ U_{m,n+1} \approx (1 + \Delta t L_1)(1 + \Delta t L_2) U_{m,n} \]  
\[ = (1 + kL_1 + kL_2 + k^2 L_1 L_2) U_{m,n}, \quad k = \Delta t \]  
\[ = U_{m,n} + kL_1 U_{m,n} + kL_2 U_{m,n} + k^2 L_1 L_2 U_{m,n} \]  

**Pure Crank–Nicholson Scheme**  
From the Burgers’ equation (1.1.2), it follows that,  
\[ L_1 = -\beta u \frac{\partial u_{m,n}}{\partial x} \]  
\[ \approx -\beta U_{m,n} \mu \delta x \]  
and  
\[ L_2 U_{m,n} = \lambda \frac{\partial^2 u_{m,n}}{\partial x^2} \]  
\[ \approx \frac{\lambda}{2\pi} \delta^2 x (U_{m,n} + U_{m,n+1}) \]  

Equations (2.2.1) and (2.2.2) can be used in equation (2.1.3) to obtain the pure Crank-Nicholson scheme.

**Hybrid Crank–Nicholson –Lax-Friedrichs’ Scheme**  
To obtain the scheme, the term \( U_{m,n} \) can be replaced by \( \frac{1}{\Delta x}(U_{m-1,n} + U_{m+1,n}) \) in equation (2.1.3) and used equations (2.2.1) and (2.2.2).

**Hybrid Crank–Nicholson–Du Fort and Frankel Scheme**  
To obtain the scheme, the term \( U_{m,n} \) is replaced by \( \frac{1}{\Delta x}(U_{m,n-1} + U_{m,n+1}) \) in the equation (2.2.2), \( U_{m,n} \) by \( \frac{1}{\Delta x} U_{m,n-1} \) and \( U_{m,n+1} \) by \( \frac{1}{\Delta x} U_{m,n+1} \) in equation (2.1.3).

**Hybrid Crank–Nicholson–Lax-Friedrich–Du Fort and Frankel Scheme**  
To obtain the scheme, the term \( U_{m,n-1} \) is replaced by \( \frac{1}{\Delta x}(U_{m-1,n-1} + U_{m+1,n+1}) \).

**Approximation at the boundaries**  
The composite operator \( L_1 L_2 U_{m,n} \) results in \( U_{m-2,n} \) and \( U_{m+2,n} \) as some of its values. They are actually values along the left and right boundaries respectively. To approximate them Von Neumann boundary conditions are used.

Wood (2006), gives the exact solution of Burgers’ equation (1.1) as  
\[ u(x,t) = \frac{2\lambda \pi e^{-x^2/\mu}}{d + e^{-x^2/\mu} \cos \pi x}, \quad d > 1 \]  
(2.6.1)  
and so  
\[ u(x,0) = \frac{2\lambda \pi \sin \pi x}{d + \cos \pi x}, \quad d > 1, \]  
(2.6.2)
\begin{equation}
    u(0,t) = u(1,0) = 0,
\end{equation}

\begin{equation}
    u_x(0,t) = \frac{2\lambda \pi^2}{d \exp(\pi^2 \lambda t) + 1},
\end{equation}

\begin{equation}
    u_x(1,t) = \frac{-2\lambda \pi^2}{d \exp(\pi^2 \lambda t) - 1}.
\end{equation}

At the left boundary (that is at \( x = 0 \)) we have

\begin{equation}
    U_{m-2,0} = U_{m-1,0} - \frac{2h\lambda \pi^2}{d \exp(-\pi^2 \lambda \omega k) + 1}\tag{2.6.6}
\end{equation}

At the right boundary (that is at \( x = 1 \)),

\begin{equation}
    U_{m+2,0} = U_{m+1,0} - \frac{2h\lambda \pi^2}{d \exp(-\pi^2 \lambda \omega k) - 1}\tag{2.6.7}
\end{equation}

In equations (2.6.6) and (2.6.7), \( \omega = n, \ n+1 \)

**Results of the numerical schemes**

**Display of the results**

Solutions of the methods developed for \( d = 2, \alpha = 0.0001 \) and \( \beta = 1 \) are generated.

The following notations are used throughout the presentations;

- CN means pure Crank-Nicholson’s method,
- CN–LF means Crank-Nicholson- Lax-Friedrich’s method,
- CN–DF means Crank-Nicholson-Du Fort–Frankel’s method and
- 3–D means three dimensional,
- OPS means operator splitting.

The following figures give the 2–D or 3–D solution of the Burgers’ equation using the various methods discussed above. In all cases, the figures are self explanatory.
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Figure 3: Fourth order operator splitting solutions of the Burgers equation from different Methods at t=0.005

Figure 4: CN 3-D solution for Burgers equation from first order operator splitting

Figure 5: CN-LF 3-D solution for Burgers equation from first order operator splitting

Figure 6: CN-DF 3-D solution for Burgers equation from first order operator splitting

Figure 7: CN 3-D solution for Burgers equation from second order operator splitting

Figure 8: CN-LF 3-D solution for Burgers equation from second order operator splitting

Figure 9: CN-DF 3-D solution for Burgers equation from second order operator splitting

Figure 10: CN-LF-DF 3-D solution for Burgers equation from second order operator splitting
Results and Discussion

Results indicate that the Lax–Friedrichs’ reduces the efficacy of the Crank-Nicholson method, the Du–Fort and Frankel differencing improves the efficacy of the Crank–Nicholson and the hybrid Crank–Nicholson-Lax–Friedrichs methods. The increase of grid points involved is responsible for the improved accuracy of the Crank–Nicholson method and the hybrid Crank–Nicholson–Lax–Friedrichs. The Du Fort and Frankel method increases the number of grid points involved by one. The hybrid Crank–Nicholson–Lax–Friedrich–Du Fort and Frankel method of the fourth order operator splitting is the most accurate and the grooves in the 3–D solution indicates that the accuracy is improved or decreased from one time value to the next.

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References

Gottlieb D. and Gustafsson B. (1996), Generalized Du Fort Franked Methods for parabolic IVPS VOL13 NO 1, SIAM NUMBER ANAL.