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Hybrid finite difference schemes from operator splitting for solving Burgers' equation

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Burgers' equation appears as a model in turbulence and gas dynamics. We develop hybrid finite difference methods resulting from operator splitting for solving it. Among the hybrid finite difference methods developed are the Crank–Nicholson–Lax–Fredrichs, Crank-Nicholson–Du Fort and Frankel and Crank–Nicholson–Lax-Fredrich–Du Fort and Frankel. We determine that Lax–Fredrichs method reduces the efficacy of the Crank–Nicholson method whereas the Du Fort and Frankel method increases the efficacy of this method (the Crank-Nicholson method). The Du–Fort and Frankel method actually increase the number of grid points involved. The increase in number of grid points used is responsible for the improved accuracy of the pure Crank–Nicholson and the hybrid Crank–Nicholson-Lax–Friedrichs' schemes. The hybrid Crank-Nicholson-Lax-Friedrichs' scheme is the most accurate.

Keywords: Burgers' equation, operator splitting, Crank-Nicholson method.

Introduction

Finite difference methods for solving the heat equation

Consider the heat equation:

$$u_t = \alpha u_{xx} \quad u(0,x) = u_0(x) \tag{1.1.1}$$

Studies by Ames (1977), Mitchell and Griffiths (1980), Jain (1984), Chapra and Canale (1998), Rahman (1994) and Rao (2005) describe the three methods Schmidt, Crank-Nicholson , Lax-Fredrich's and Du Fort and Frankel methods for finding the numerical solution of the equation (1.1.1).These methods are based on finite differences. Schmidt, Lax–Fredrichs' and Du Fort and Frankel methods are explicit whereas the Crank–Nicholson method is implicit. Gotlieb and Gustafson in their paper provide a thorough analysis of the Du Fort and Frankel method.

Schmidt and Lax-Fredrichs' methods are conditionally stable whereas Crank-Nicholson and Du Fort and Frankel are unconditionally stable.

In our paper we develop blended (hybrid) finite difference methods for finding the numerical solution of the Burgers' equation

$$u_t + \beta uux = \alpha u_{xx} \quad (0 \le x \le 1) \times (t \ge 0) \tag{1.1.2}$$

$u(0,x) = u_0(x)$
$u_x(0,t) = p(x)$
$u_x(1,t) = q(x)$
resulting from the operator splitting.

Overview of Operator Splitting

The operator splitting technique for the linear parabolic equation is outlined, thus:

$$u_t = Lu, \quad (a \le x \le b) \times (0 \le t \le T) \tag{1.2.1}$$

$$u(x,0) = u_0(x) \tag{1.2.2}$$

where $u = u(x,t)$ and L is a linear differential operator.

Consider the Taylor's expansion

$$u(x,t+k) = u(x,t) + k\frac{\partial}{\partial t}(u(x,t)) + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2}(u(x,t)) + \dots$$

$$= \left(1 + k\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2} + \dots\right)u(x,t)$$

$$= e^{k\frac{\partial}{\partial t}}(u(x,t)) \tag{1.2.3}$$

In equation (1.2.3) $\frac{\partial}{\partial t}$ can be replaced by L

that is

$$u(x,t+k) = e^{kL}(u(x,t)) \tag{1.2.4}$$

The exact solution of the equation (1.2.1)-(1.2.2) at the grid point ($x = mh, t = nk$) is $u(x,t)$. The approximate solution at this point is denoted by $U_{m,n}$. Equation (1.2.4) can be written as;

$$U_{m,n+1} = e^{kL}U_{m,n} \tag{1.2.5}$$

In equations (1.4) and (1.2.5) e^{kL} is called the solution operator for equation (1.2.1) L is replaced by finite difference approximation. In equation (1.2.5) L can be taken to be a sum of differential operators with respect to x. If

$$L = L_1 + L_2 + L_3 + \dots\dots + L_S = \sum_{i=1}^{S} L_i$$

then equation (1.2.5) cab be written as

$$U_{m,n} = e^{k\sum_{i=1}^{S} L_i} U_{m,n}$$

$$= (e^{k(L_1+L_2+L_3+\dots+L_S)})U_{m,n} \tag{1.2.6}$$

$$= (e^{kL_1}(e^{kL_2}(....(e^{kL_{S-1}}(e^{kL_S}U_{m,n}))....))) \tag{1.2.7}$$

$$= \prod_{i=1}^{S} e^{kL_i} U_{m,n}$$

The approximate solution can be obtained from equation (1.2.7) by first solving $U^{(S)}_{m_1 n+1} = e^{kL_S} U_{m,n}$ and then using this solution it is found that $U^{(S-1)}_{m,n+1} = e^{L_{S-1}} U_{m,n}$
This continues until $U^{(1)}_{m,n+1}$ is obtained which is actually the approximate solution of equation (1.2.1) (see Istvan, 2003).

For the Burgers equation

$$u_t + \beta u u_x = \alpha u_{xx} \tag{1.2.8}$$

The number of operators are two; that is S=2 with

$$L_1 = -\tfrac{\beta}{h} U_{m,n} \delta_x \tag{1.2.9}$$

and

$$L_2 = \tfrac{\alpha}{h^2} \delta_x^2 \tag{1.2.10}$$

The discretizations of L_1 and L_2 can now be done using various ways.

Historical background of operator splitting techniques can now be provided. Koller and Krylov (2006) demonstrated and discussed time integration due to operator splitting for linear 1-D parabolic equations. Ames (1977) and Mitchell and Griffiths (1980) describes additive operator splitting for parabolic equations which are more than one dimensional and were developed by Yanenko and Marchuk. Yanenko splitting is called first order operator splitting. Another splitting method mentioned by the same Mitchell and Griffiths (1980) which is called second order was developed by Strang in the 1960s. Istvan gives an elaborate discussion of operator splitting for parabolic eaquations. Le Veque and Oliger (1983) describes additive operator splitting for hyperbolic partial differential equations. Splitting method has been used by Hvistendahl (1997) and Evje *et al.,* (1998) to find the numerical solution of convection–diffusion equation.

In the study, a hybrid finite difference schemes based on Yanenko and Marchuk ideas to solve the nonlinear Burgers' equation was developed. Solution of the heat equation were found and discussed.

Below, hybrid finite difference schemes resulting from first order operator splitting are developed.

Hybrid Finite Difference Schemes from Operator Splitting

The word hybrid means blended or "marrying" two or more ordinary finite difference methods. To begin with, it is necessary to develop the pure Crank-Nicholson scheme as follows:

First order operator splitting

The first order operator splitting numerical solution is given by

$$U_{m,n+1} = \left(e^{\Delta t L_1} e^{\Delta t L_2}\right) U_{m,n} \tag{2.1.1}$$

where L_1 and L_2 are the operators stated above, then equation (2.1.1)

$$U_{m,n+1} \approx (1+\Delta t L_1)(1+\Delta t L_2) U_{m,n} \tag{2.1.2}$$

$$= (1 + kL_1 + kL_2 + k^2 L_1 L_2) U_{m,n}, \ k = \Delta t$$

$$= U_{m,n} + kL_1 U_{m,n} + kL_2 U_{m,n} + k^2 L_1 L_2 U_{m,n} \tag{2.1.3}$$

Pure Crank–Nicholson Scheme

From the Burgers' equation (1.1.2), it follows that,

$$\begin{aligned} L_1 &= -\beta u \tfrac{\partial u_{m,n}}{\partial x} \\ &\approx -\beta U_{m,n} \mu \delta_x \end{aligned} \tag{2.2.1}$$

and

$$\begin{aligned} L_2 U_{m,n} &= \lambda \tfrac{\partial^2 u_{m,n}}{\partial x^2} \\ &\approx \tfrac{\lambda}{2h^2} \delta_x^2 (U_{m,n} + U_{m,n+1}) \end{aligned} \tag{2.2.2}$$

Equations (2.2.1) and (2.2.2) can be used in equation (2.1.3) to obtain the pure Crank-Nicholson scheme.

Hybrid Crank–Nicholson –Lax-Friedrichs' Scheme

To obtain the scheme, the term $U_{m,n}$ can be replaced by $\tfrac{1}{2}(U_{m-1,n} + U_{m+1,n})$ in equation (2.1.3) and used equations (2.2.1) and (2.2.2).

Hybrid Crank–Nicholson–Du Fort and Frankel Scheme

To obtain the scheme, the term $U_{m,n}$ is replaced by $\tfrac{1}{2}(U_{m,n-1} + U_{m,n+1})$ in the equation (2.2.2), $U_{m,n}$ by $\tfrac{1}{2}U_{m,n-1}$ and $U_{m,n+1}$ by $\tfrac{1}{2}U_{m,n+1}$ in equation (2.1.3).

Hybrid Crank–Nicholson–Lax-Friedrich–Du Fort and Frankel Scheme

To obtain the scheme, the term $U_{m,n-1}$ is replaced by $\tfrac{1}{2}(U_{m-1,n-1} + U_{m+1,n+1})$.

Approximation at the boundaries

The composite operator $L_1 L_2 U_{m,n}$ results in $U_{m-2,.}$ and $U_{m+2,.}$ as some of its values. They are actually values along the left and right boundaries respectively. To approximate them Von Neumann boundary conditions are used.

Wood (2006), gives the exact solution of Burgers' equation (1.1) as

$$u(x,t) = \frac{2\lambda \pi e^{-\pi^2 \lambda t} \sin \pi x}{d + e^{-\pi^2 \lambda t} \cos \pi x}, \quad d > 1 \tag{2.6.1}$$

and so

$$u(x,0) = \frac{2\lambda \pi \sin \pi x}{d + \cos \pi x}, \ d > 1, \tag{2.6.2}$$

$$u(0,t) = u(1,0) = 0\,, \tag{2.6.3}$$

$$u_x(0,t) = \frac{2\lambda\pi^2}{d\exp(\pi^2\lambda t)+1}, \tag{2.6.4}$$

$$u_x(1,t) = \frac{-2\lambda\pi^2}{d\exp(\pi^2\lambda t)-1} \tag{2.6.5}$$

At the left boundary (that is at $x = 0$) we have

$$U_{m-2,\omega} = U_{m-1,\omega} - \frac{2h\lambda\pi^2}{d\exp(-\pi^2\lambda\omega k)+1} \tag{2.6.6}$$

At the right boundary (that is at $x = 1$),

$$U_{m+2,\omega} = U_{m+1,\omega} - \frac{2h\lambda\pi^2}{d\exp(-\pi^2\lambda\omega k)-1} \tag{2.6.7}$$

In equations (2.6.6) and (2.6.7), $\omega = n,\ n+1$

Results of the numerical schemes

Display of the results

Solutions of the methods developed for $d = 2, \alpha = 0.0001$ and $\beta = 1.$ are generated.

The following notations are used throughout the presentations;
CN means pure Crank-Nicholson's method,
CN–LF means Crank-Nicholson- Lax-Friedrich's method,
CN–DF means Crank-Nicholson-Du Fort–Frankel's method and
CN–LF–DF means Crank-Nicholson- Lax–Friedrich–Du Fort–Frankel's method,
3–D means three dimensional,
OPS means operator splitting.

The following figures give the 2–D or 3–D solution of the Burgers' equation using the various methods discussed above. In all cases, the figures are self explanatory.

Figure 1:Solutions of the Burgers Equation from first order operator splitting methods at t=0.005

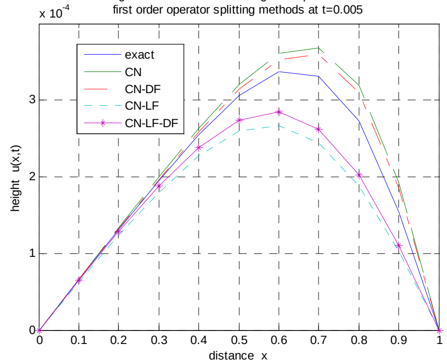


Figure 2: Second order operator splitting solutions of the Burgers equation from different methods at t=0.005

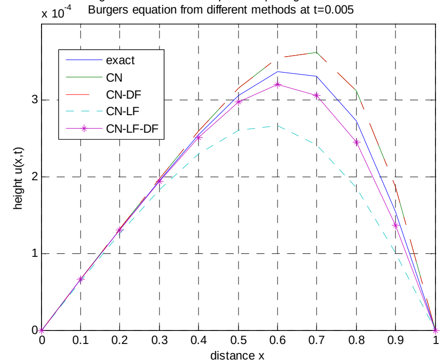


Figure 3: Fourth order operator splitting solutions of the Burgers equation from different Methods at t=0.005

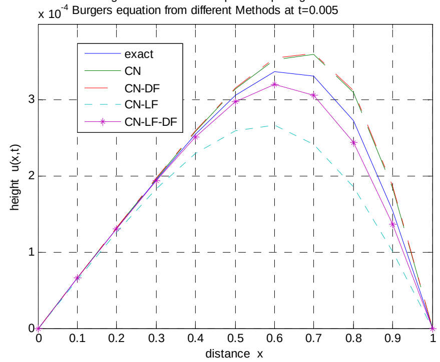


Figure 4: CN 3-D solution for Burgers equation from first order operator splitting

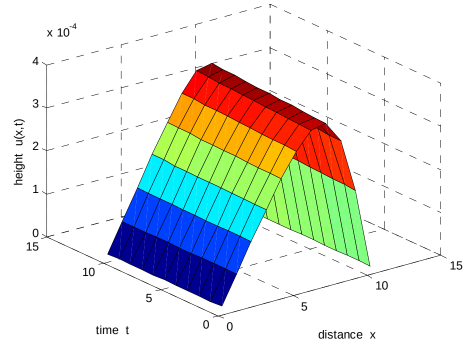


Figure 5: CN-LF 3-D solution for Burgers equation from first order operator splitting

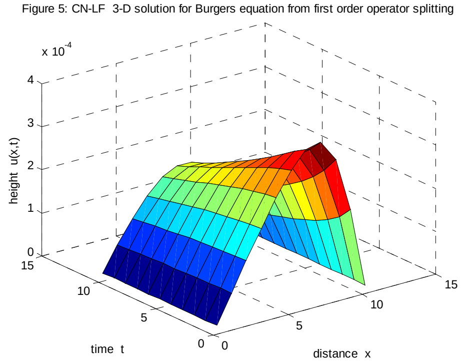


Figure 6 :CN-DF 3-D solution for Burgers equation from first order operator splitting

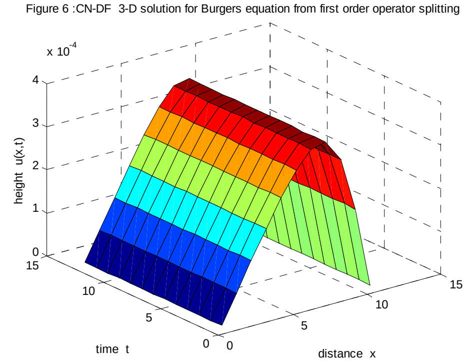


Figure 8:CN-LF 3-Dsolution for Burgers equation from second order operator splitting

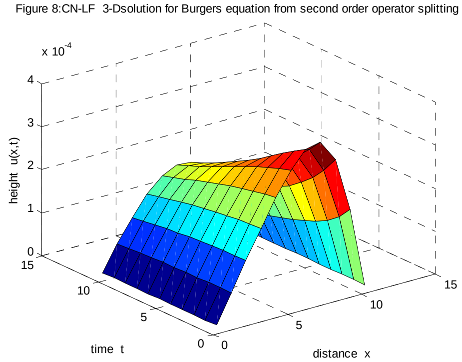


Figure 7:CN 3-D solution for Burgers equation from second order operator splitting

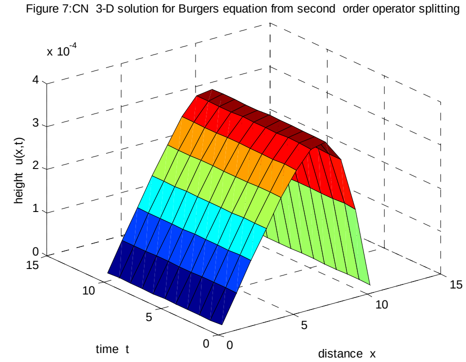


Figure 9:CN-DF 3-D solution for Burgers equation from second order operator splitting

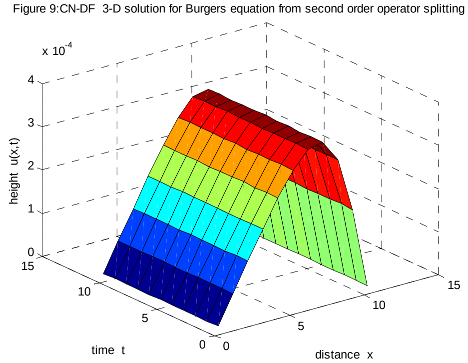


Figure 10 :CN-LF-DF 3-D solution for Burgers equation from second order operator splitting

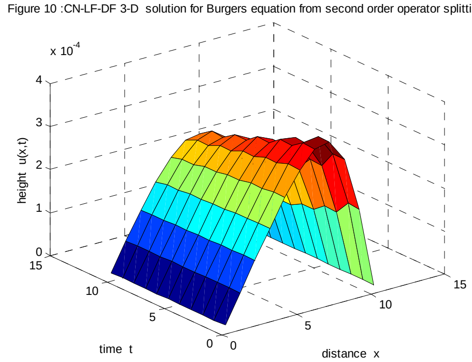


Figure 11:CN 3-D solution for Burgers equation from fourth order operator splitting

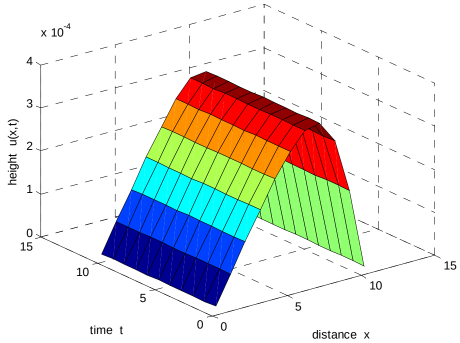


Figure 12 :CN-LF 3-D solution for Burgers equation from fourth order operator splitting

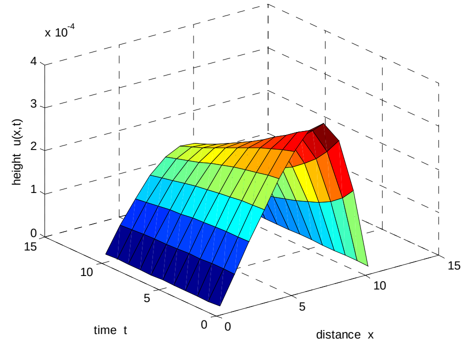


Figure 13 :CN-DF 3-D solution for Burgers equation from fourth order operator splitting

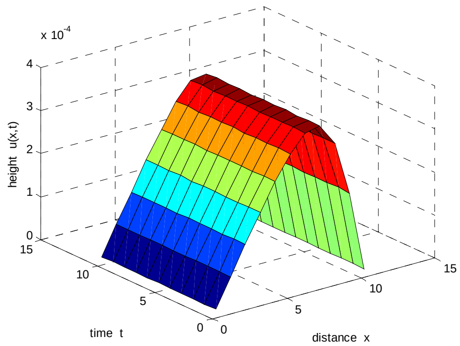
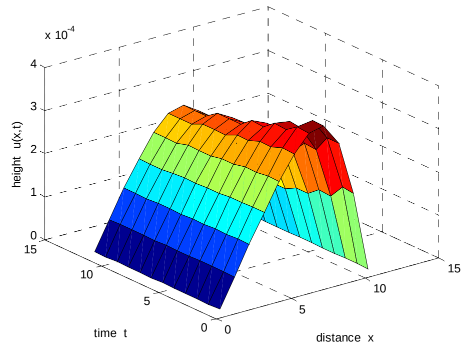


Figure 14: CN-LF-DF 3-D solution for Burgers equation from ordinary methods



It is noted that the 3-D solutions that do not involve the Du Fort and Frankel method are smooth wheres those that involve it are grooved.

Results and Discussion

Results indicate that the Lax–Friedrichs' reduces the efficacy of the Crank-Nicholson method, the Du–Fort and Frankel differencing improves the efficacy of the Crank–Nicholson and the hybrid Crank–Nicholson-Lax–Friedrichs methods. The increase of grid points involved is responsible for the improved accuracy of the Crank–Nicholson method and the hybrid Crank–Nicholson–Lax–Friedrichs. The Du Fort and Frankel method increases the number of grid points involved by one. The hybrid Crank–Nicholson-Lax–Friedrich–Du Fort and Frankel method of the fourth order operator splitting is the most accurate and the grooves in the 3–D solution indicates that the accuracy is improved or decreased from one time value to the next.

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